High Energy String Theory and the Celestial Sphere

Connecting worldsheet CFT and celestial CFT

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Introduction

- String theory is equipped with an intrinsic parameter, α'. In the field theory limit ($\alpha' \to 0$), strings become point-like and describe a QFT. The opposite limit, $\alpha' \to \infty$, leads to the appearance of an infinite number of massless higher-spin states in the string spectrum [1]. Moreover, most of the powerful properties of string amplitudes are known to stem from the inherent 2d CFT structure on the string worldsheet: soft UV behaviour, modular invariance and loop corrections, etc.
- A 2d CFT also arises in the celestial holography program, where 4d scattering amplitudes in asymptotically flat spacetimes are recast as conformal correlators on the celestial sphere at null infinity, $\mathcal{I}$. The resulting *celestial amplitudes* are tightly constrained by the symmetries of the underlying celestial CFT (CCFT), which is expected to eventually allow for first-principles computations of bulk physics from boundary data, similar to the well-known AdS/CFT correspondence.
- Since celestial amplitudes are obtained via Mellin transforms that integrate out energy, they are typically UV divergent in field theory–unlike in string theory, where UV-softness ensures well-defined celestial string amplitudes [2]. We exploit this feature in our paper to relate the free worldsheet CFT to the CCFT in the high-energy limit of string theory.
- Specifically, we relate the saddle-point approximation of string amplitudes as $\alpha' \to \infty$ with a stationary phase expansion of celestial string amplitudes for large conformal weight, at all orders. This approach points to an intrinsic construction of CCFT by relating it to a (free) worldsheet CFT.

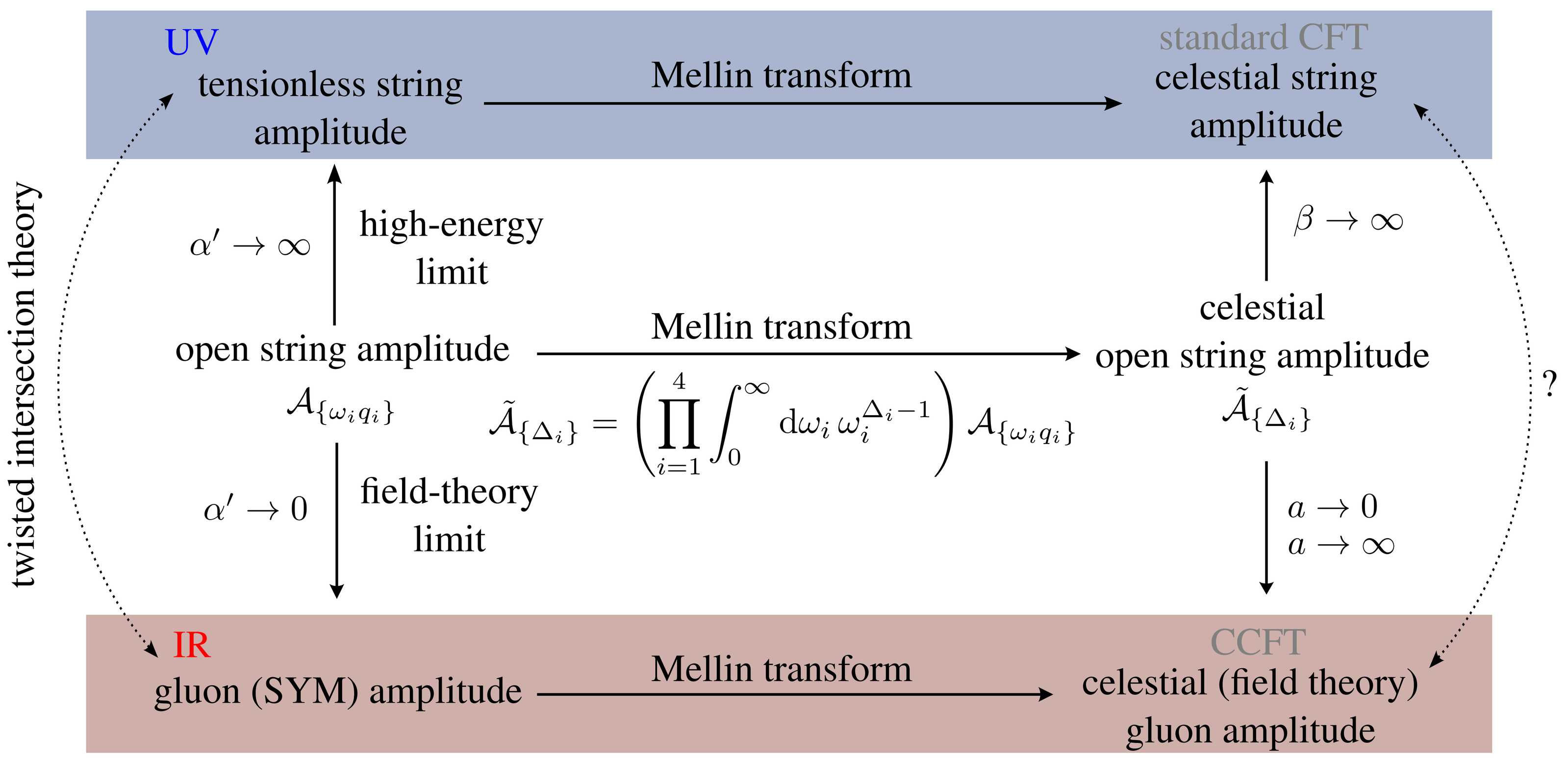

Figure 1: Various limits of the 4-point open string gluon amplitude considered in this work.

Representations of string amplitudes

For canonical colour ordering, the 4-point tree-level open string (gluon) amplitude is given by

$$\mathcal{A}(1,2,3,4) = \frac{\Gamma(1-s)\Gamma(1-u)}{\Gamma(1+t)}A_{\text{YM}}(1,2,3,4) = -A_{\text{YM}}(1,2,3,4)s\int_0^1 \mathrm{d}x\, x^{-s-1}(1-x)^{-u}, \quad (1)$$

with the Mandelstam invariants $s = \alpha'(p_1+p_2)^2$, $t = \alpha'(p_1-p_3)^2$ $(s+t+u=0)$. For closed strings,

$$\mathcal{M} = \pi\frac{su}{t}\frac{\Gamma(-s)\Gamma(-u)\Gamma(-t)}{\Gamma(s)\Gamma(u)\Gamma(t)}A_{\text{YM}}(1,2,3,4)\tilde{A}_{\text{YM}}(1,2,3,4).$$

In the field theory ($\alpha' \to 0$) limit, one has [3]

$$\mathcal{A}_0 = \exp\left\{\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)}\left(s^{2n}+u^{2n}-t^{2n}\right)\right\}\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(2k+1)}\left(s^{2k+1}+u^{2k+1}+t^{2k+1}\right)\right\}A_{\text{YM}}.$$

When $\alpha' \to \infty$ on the other hand, we find

$$\mathcal{A}_{+\infty} = \sqrt{2\pi\frac{su}{t}\frac{\sin(\pi t)}{\sin(\pi s)}}\,s^{-s}u^{-u}t^{-t}(-1)^{-u-t}\underbrace{\exp\left\{\sum_{k=1}^{\infty}\frac{\zeta(1-2k)}{(2k-1)}\left(\frac{1}{s^{2k-1}}+\frac{1}{u^{2k-1}}+\frac{1}{t^{2k-1}}\right)\right\}}_{(\bigstar)}A_{\text{YM}}.$$

This expression may be understood as quantum fluctuations around the classical solution $\mathcal{A}_c$ to the path integral

$$\mathcal{A} \sim \int \mathcal{D}g\,\mathcal{D}X\,\exp\left\{-\frac{1}{4\pi\alpha'}\int \mathrm{d}\sigma^1\mathrm{d}\sigma^2\sqrt{g}g^{\alpha\beta}\partial_\alpha X^\mu\partial_\beta X_\mu\right\}\prod_{i=1}^{4}V_o(p_i) \overset{\alpha'\to\infty}{\sim} \mathcal{A}_c,$$

with the positions z_i of the four open string vertices V_o subject to the condition

$$x_0 \equiv \frac{(z_1-z_2)(z_3-z_4)}{(z_1-z_3)(z_2-z_4)} = -\frac{s}{t}. \quad (2)$$

This amounts to performing the saddle-point approximation of (1) around x_0,

$$-s\int_0^1 \mathrm{d}x\, x^{-s-1}(1-x)^{-u} = \sqrt{\frac{2\pi as}{1-a}}B^s\left[1+\sum_{n=1}^{\infty}\frac{C_{2n}}{(-s)^n}\right], \quad C_{2l} = (\bigstar)\big|_{\frac{1}{s^l}}, \quad a = -\frac{u}{s}, \quad (3)$$

where $B = (1-a)^{1-a}(-a)^a$. Subleading terms are suppressed in powers of $\frac{1}{\alpha'}$ and encode the effects of integrating out massive higher-spin states at high energies. In our paper, we repeat those steps for the Virasoro-Shapiro amplitude of closed strings and find that the single-value map [4]

$$\text{sv}\zeta(2k) = 0, \quad \text{sv}\zeta(2k+1) = 2\zeta(2k+1), \quad k \geq 1$$

is only valid in the low-energy regime ($\alpha' \to 0$), where one has

$$\mathcal{M}_0 = \pi\,\text{sv}\mathcal{A}_0(1,2,3,4)\,\tilde{A}_{\text{YM}}(1,2,3,4).$$

String worldsheet and celestial sphere

Parametrizing massless momenta as $p^\mu = \omega q^\mu$, $q^\mu = \frac{1}{2}(1+|z|^2, z+\bar{z}, -i(z-\bar{z}), 1-|z|^2)$, with $z_i, \bar{z}_i$ coordinates at $\mathcal{I}$, the celestial string amplitude is obtained as

$$\tilde{\mathcal{A}}_{\{\Delta_l\}}(\{z_l,\bar{z}_l\}) = \left(\prod_{k=1}^{n}\int_0^\infty \mathrm{d}\omega_k\,\omega_k^{\Delta_k-1}\right)\delta^{(4)}\left(\omega_1 q_1+\omega_2 q_2-\sum_{m=3}^{n}\omega_m q_m\right)\mathcal{A}(\{\omega_l,z_l,\bar{z}_l\}).$$

For (1) this yields, with $\tilde{\mathcal{A}}'_{\text{FT}}$ the field-theoretic celestial gluon amplitude [2]

$$\tilde{\mathcal{A}}_{\{\Delta_i\}} = (2\pi)^{-1}(\alpha')^\beta\tilde{\mathcal{A}}'_{\text{FT}}(\{\Delta_i\})a^{-\frac{\beta}{3}}(1-a)^{-\frac{\beta}{3}}I(a,\beta), \quad \beta \equiv -\frac{1}{2}\sum_{k=1}^{4}(\Delta_k-1), \quad (4)$$

$$\text{with } I(a,\beta) = -\Gamma(1-\beta)\int_0^1\frac{\mathrm{d}x}{x}\left[\ln x - a\ln(1-x)\right]^{\beta-1}, \quad (5)$$

which has poles for $\beta = 1,2,3,\ldots$ As $a \to 0$ or ∞, $\tilde{\mathcal{A}} \to \tilde{\mathcal{A}}'_{\text{FT}}$, as one may write [2]

$$I(a,\beta) = \pi\delta(\beta) + \frac{i\pi}{1-e^{-2\pi i\beta}}\sum_{n=1}^{\infty}\text{Res}_{s=n}\left[(as)^{-\beta}B\left(-s,1+\frac{s}{r}\right)\right].$$

Thus, $\beta = n \in \mathbb{N}^*$ is a soft pole of (5) associated with the exchange of massive strings at level n. Here we perform a stationary phase approximation of $I(a,\beta)$ as $\beta \to \pm i\infty$ instead, keeping a fixed. This also localizes on the saddle (2), meaning that the worldsheet pins onto the celestial sphere as $\alpha' \to \infty$. We then match this expansion in $\frac{1}{\beta}$ with the Mellin transform of (3), and find

$$B^{\alpha'}\frac{C_{2k}}{(\alpha')^k} \leftrightarrow (\alpha')^\beta\frac{(-1)^k C_{2k}(\ln B)^{\beta-\frac{1}{2}+k}}{\cos(\pi\beta)\Gamma\left(\beta+\frac{1}{2}+k\right)} \sim (\alpha')^k\frac{(-1)^k C_{2k}(\ln B)^{\beta-\frac{1}{2}+k}}{\left(\beta+\frac{1}{2}\right)\cdot\ldots\cdot\left(\beta-\frac{1}{2}+k\right)}. \quad (6)$$

This shows that the poles at $\beta = -\frac{1}{2}, -\frac{3}{2}, \ldots$ of (5) are in 1:1 correspondence with the subleading corrections in $\frac{1}{(\alpha')^k}$ in (3). In the β-plane these effects correspond to operators with $\Delta_k \geq 2$ and account for higher-spin modes in the ultra-high energy regime of string theory. From (3) we see that these are tied to combinations of $\zeta(1-2k) \in \mathbb{Q}$. Finally, $\Gamma(1-\beta) \propto |\beta|^{|\beta|}$ as $\Re(\beta) \to -\infty$, which is a manifestation of black hole dominance [5].

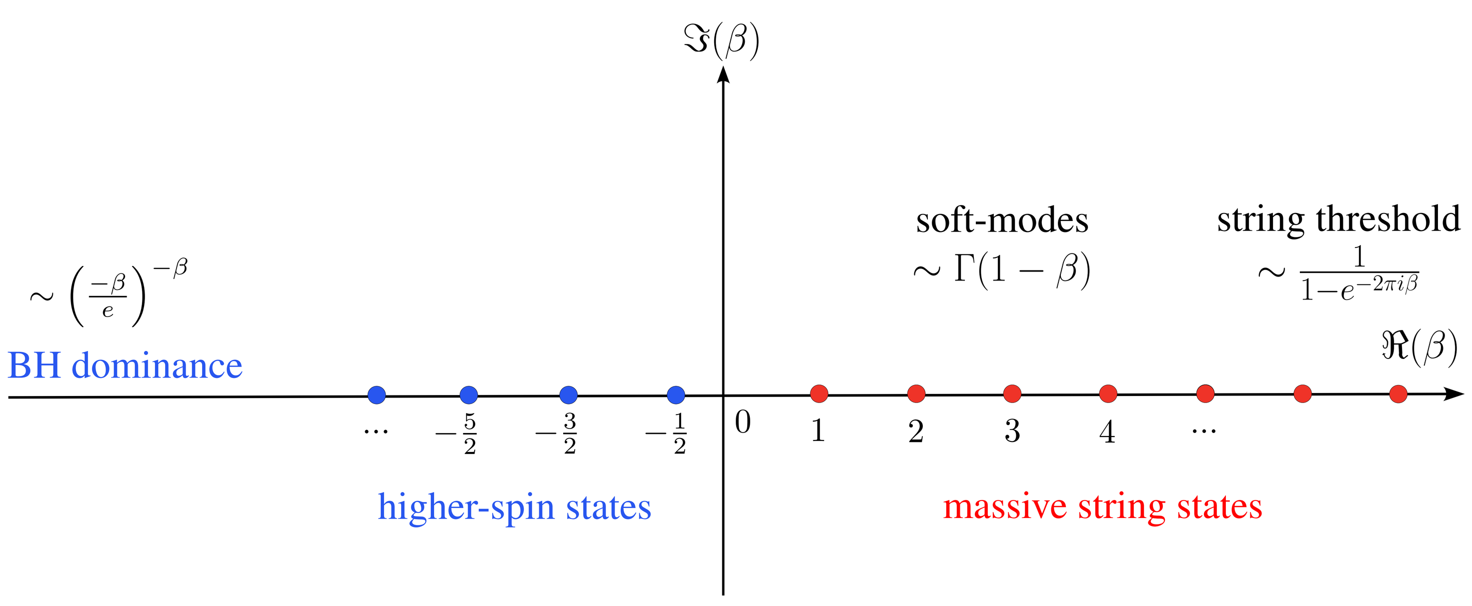

Figure 2: Complex β plane with UV (blue) and IR (red) regions and corresponding string threshold.

Conclusion

We have presented expressions for string amplitudes in the high-energy limit at tree level in flat backgrounds, highlighting their number-theoretic properties, and showing that the sv-map is only valid as $\alpha' \to 0$. We have also established the correspondence (6) between the celestial open string amplitude and the string amplitude as $\alpha' \to \infty$, order by order in $\frac{1}{\alpha'}$ and $\frac{1}{\beta}$. This is a first step towards a string realization of flat-space holography, relating the worldsheet CFT to the (less understood) CCFT.

Open questions, in no particular order:

- Generalization to $n \geq 5$? 1-loop? Define generic conditions for bijective map with CCFT?
- Could studying the $\alpha' \to \infty$ and $\beta \to -\infty$ limits shed light on properties of CCFT in the IR?
- $\alpha' \to \infty \sim$ tensionless strings, with $\text{CCS}_2 \cong \text{BMS}_3$ symmetry [6]. Lessons from CCFT?
- Twisted intersection theory relates $\alpha' \to 0$ and $\alpha' \to \infty$. What about the celestial side?
- Closed strings and KLT/double-copy relations? String monodromy on the celestial sphere?
- Relation to results in AdS? Cf, e.g., the work of Alday et al. [7]
- What is the Carrollian analogue of this picture?

References

[1] David J. Gross and Paul F. Mende. String Theory Beyond the Planck Scale. *Nucl. Phys. B*, 303:407–454, 1988.

[2] Stephan Stieberger and Tomasz R. Taylor. Strings on Celestial Sphere. *Nucl. Phys. B*, 935:388–411, 2018.

[3] O. Schlotterer and S. Stieberger. Motivic Multiple Zeta Values and Superstring Amplitudes. *J. Phys. A*, 46:475401, 2013.

[4] Stephan Stieberger and Tomasz R. Taylor. Closed String Amplitudes as Single-Valued Open String Amplitudes. *Nucl. Phys. B*, 881:269–287, 2014.

[5] Nima Arkani-Hamed, Monica Pate, Ana-Maria Raclariu, and Andrew Strominger. Celestial amplitudes from UV to IR. *JHEP*, 08:062, 2021.

[6] Arjun Bagchi. Tensionless Strings and Galilean Conformal Algebra. *JHEP*, 05:141, 2013.

[7] Luis F. Alday, Tobias Hansen, and Maria Nocchi. High Energy String Scattering in AdS. *JHEP*, 02:089, 2024.